JOURNAL OF COMPUTATIONAL PHYSICS 56, 149–164 (1984)

Splines and a Three-Body Separable Expansion for Scattering Problems

D. EYRE

National Research Institute for Mathematical Sciences, CSIR, P.O. Box 395, Pretoria 0001, Republic of South Africa

Received June 8, 1983; revised February 13, 1984

This paper investigates an energy-dependant finite rank approximation to the three-body amplitude at scattering energies both below and above the breakup threshold. The expansion method is one proposed by Adhikari and Sloan as an approximation method for solving few-body integral equations. Cubic spline functions are used to evaluate the expansion terms without contour rotation. Numerical results are obtained for a system of three identical bosons and the utility of this expansion method in a numerical treatment of the integral equations for four-body scattering is briefly discussed. Due to logarithmic singularities in the off-shell three-body amplitude at positive energies, no pointwise agreement is found between the separable expansion and the exact three-body amplitude at energies above the breakup threshold.

I. INTRODUCTION

In the work of Faddeev [1] and Yakubovskii [2], among others [3], it has been shown that the few-body problem can be reduced to that of solving multidimensional integral equations. A further useful reduction of this problem can be achieved by introducing a separable expansion method [3] to the subsystem amplitudes. For example, approximating the two-particle amplitude by a sum of separable terms will, after a partial wave decomposition, reduce the Faddeev equations to a set of coupled integral equations in one continuous momentum variable. The utility of such a method is determined by the number of separable terms needed for an accurate solution to the resulting integral equations.

A reduction of the three-body problem to a two-body Lippmann–Schwinger-type equation [4], brought about by the use of separable potentials, is now well understood. A similar reduction of the four-body problem by introducing a separable expansion to the three-body amplitude can be achieved in several ways. One approach is to apply the Hilbert–Schmidt method [5]. Another approach which uses an energy-dependent pole expansion has proved successful in the case of bound state calculations [6]. But as yet a practical application of this expansion approximation to four-body scattering has not been fully investigated [7]. A difficulty here is the singular behaviour of the three-body amplitude at positive energies [8]. In addition to

0021-9991/84 $3.00

a non-smooth (cusp) behaviour, the off-energy-shell three-body amplitude contains logarithmic singularities.

In the present paper we investigate an expansion method described by Adhikari and Sloan [9] and later used by Sofianos *et al.* [6] in their four-body bound state calculations. We place our emphasis on scattering problems and in particular the utility of the approximate three-body amplitude in the context of four-body scattering equations. While Sofianos *et al.* have demonstrated the success of their expansion method for the three-body bound states, the current paper is devoted to an investigation of this expansion method in the three-body scattering sector.

A novel feature of our approach to constructing the Adhikari–Sloan expansion is that we evaluate the three-body solution along the real energy axis. A numerical treatment by Fonseca *et al.* [7] used the method of contour rotation. A difficulty with this technique is that prior knowledge is required of the analytic structure of the three-body amplitude in the complex momentum plane [10]. For this reason we propose an alternative numerical technique in which cubic B-splines [11] are used to construct the expansion terms.

Sections II and III give an overview of the three-body separable expansion. Section II gives a formal operator description of the equations and Section III describes the structure of these equations in momentum space. Section IV describes our numerical treatment of the separable expansion using spline functions. Finally, Section V gives our numerical results for a model calculation on the three-boson problem at energies both below and above the breakup threshold. The solution to the two-particle problem, which provided input to the three-body equations, is given in Appendix A. Appendix B describes the Galerkin method for solving an eigenvalue problem.

II. Three-Body Separable Expansion

This section gives a formal description of the separable expansion as applied to the three-body problem. In order to make this approximation explicit we shall consider the simple example of three identical bosons interacting via separable pairwise potentials. It will be assumed that the interaction can support a bound state in the two-particle subsystem.

For our simple example the Faddeev equations reduce [3, 4] to a two-body operator equation of the Lippmann–Schwinger type. After correct symmetrization for bosons (see (Lovelace [4]) this equation can be written

$$T(z) = U(z) - U(z)\, D(z)\, T(z) \tag{2.1a}$$

$$= U(z) - T(z)\, D(z)\, U(z), \tag{2.1b}$$

where z is a complex three-body energy. Each operator acts in a two-body space describing the motion of one particle relative to a bound state of the other two. The two-body operator $T(z)$ yields a complete description of the three-body problem.

Since the Adhikari–Sloan expansion can be applied to equations of the general type given by Eq. (2.1), a discussion of the explicit kernel form of this equation, and in particular the complicated singularity structure of the functions U and D, is deferred to the next section.

We now concentrate on the essential features of the Adhikari–Sloan expansion for Eq. (2.1). Our task is to construct a finite rank approximation to the operator $T(z)$. This is most easily done if the effective potential operator $U(z)$ can be approximated by a degenerate form. Let $U_m(z)$ be the approximation to $U(z)$ where [5]

$$U_m(z) = \sum_{\mu,\nu=1}^{m} |\psi_\mu(z) > \Lambda_{\mu\nu}(z) < \psi_\nu(z^*)|. \tag{2.2}$$

It follows, by substituting $U_m(z)$ for $U(z)$ in Eq. (2.1), that we can construct an approximation $T_m(z)$ to $T(z)$ such that

$$T_m(z) = \sum_{\mu,\nu=1}^{m} |\psi_\mu(z) > H_{\mu\nu}(z) < \psi_\nu(z^*)|, \tag{2.3}$$

where the matrix $H_{\mu\nu}(z)$ is given by

$$(H(z)^{-1})_{\mu\nu} = (\Lambda(z)^{-1})_{\mu\nu} + \langle \psi_\mu(z^*) | D(z) | \psi_\nu(z) \rangle. \tag{2.4}$$

Next we consider the functions $\{\psi_\mu; \mu = 1,..., m\}$. Let $\{f_\mu; \mu = 1,..., m\}$ be a smooth set of functions from which we construct

$$|\psi_\mu(z)\rangle = U(z)|f_\mu\rangle. \tag{2.5}$$

In this way the functions $\{\psi_\mu; \mu = 1,..., m\}$ have a built-in z-dependence of the exact effective potential operator $U(z)$. Equation (2.5) is one choice of expansion functions proposed by Adhikari and Sloan.

Returning now to Eq. (2.2), we choose the matrix $\Lambda_{\mu\nu}(z)$ so that [9]

$$(\Lambda(z)^{-1})_{\mu\nu} = \langle \psi_\mu(z^*) | f_\nu \rangle. \tag{2.6}$$

With this choice, $U_m(z)$ has the property that it is exact when operating on any linear combination of the functions $\{f_\mu; \mu = 1,..., m\}$. This follows from the fact that

$$U_m(z)|f_\mu\rangle = U(z)|f_\mu\rangle. \tag{2.7}$$

We now construct an analog to Eq. (2.7) for the operator $T_m(z)$. Using Eq. (2.1b), we write

$$T_m(z) = \sum_{\mu,\nu=1}^{m} T(z)|h_\mu(z)\rangle H_{\mu\nu}(z) \langle \psi_\nu(z^*)|. \tag{2.8}$$

where

$$(H(z)^{-1})_{\mu\nu} = \langle \psi_\mu(z^*) | h_\nu(z) \rangle, \tag{2.9}$$

and the functions $\{h_\mu ; \mu = 1,..., m\}$ are defined by the equation

$$|h_\mu(z)\rangle = [1 + D(z)\, U(z)]\,|f_\mu\rangle. \tag{2.10}$$

It follows from Eqs. (2.8) and (2.9) that $T_m(z)$ satisfies the equality

$$T_m(z)\,|h_\mu(z)\rangle = T(z)\,|h_\mu(z)\rangle. \tag{2.11}$$

It is an important feature of the approximation that $T_m(z)$ approaches $T(z)$ in the above sense.

This concludes our formal description of the Adhikari–Sloan expansion. It now remains to choose the functions $\{f_\mu ; \mu = 1,..., m\}$. Our choice is that used by Sofianos *et al.* [6] in four-body bound state calculations. This choice is known as the "energy-dependent pole expansion" (EDPE).

In order to construct the EDPE let us consider the homogeneous equation

$$-K(z)\,|\phi(z)\rangle = U(z)\, D(z)\,|\phi(z)\rangle. \tag{2.12}$$

In general there will be an infinite sequence of numbers $\{K_j(z)\}$ and functions $\{\phi_j(z)\}$ that satisfy this equation. The idea of the EDPE is to utilize a finite number m of the Sturmian functions $\{\phi_\mu(z); \mu = 1,..., m\}$ that satisfy Eq. (2.12) for the m largest eigenvalues $\{K_\mu(z); \mu = 1,..., m\}$. In particular, if the three-body system has a bound state with energy b, then we fix $z = -b$ in Eq. (2.12), solve this equation for the functions $\{\phi_\mu(-b); \mu = 1,..., m\}$, and define

$$|f_\mu\rangle = D(-b)\,|\phi_\mu(-b)\rangle. \tag{2.13}$$

It is assumed that the unit eigenvalue is contained in the sequence $\{K_\mu(-b); \mu = 1,..., m\}$. In this way the approximate operator $T_m(z)$ is forced to exhibit the correct pole behaviour at $z = -b$. The choice of functions $\{f_\mu ; \mu = 1,..., m\}$ in Eq. (2.13) is simple to evaluate and does not depend on the energy parameter z. (For some other choices based on the use of Sturmians see Ref. [8].)

III. Momentum Representation of the Expansion Functions

This section gives the explicit kernel form of Eq. (2.1) in a momentum representation. We also provide a description of the expansion functions in momentum space. For simplicity we restrict our discussion to s-wave interactions and the total angular momentum $j = 0$ case. We set the particle mass $M = 1$. In general, the variable p will denote the magnitude of the momentum of one particle with respect to the remaining pair. We reserve k for the on-shell value of this momentum. Our normalizations are chosen so that the s-wave phase shift, $\delta(k)$, is given by

$$T(k, k; E + i0) = -\frac{2}{\pi} e^{i\delta(k)} \frac{\sin \delta(k)}{k}. \tag{3.1}$$

Here we have set $z = E + i0$ where E is the three-body scattering energy

$$E = \tfrac{3}{4}k^2 - \varepsilon, \tag{3.2}$$

and ε is the two-particle bound state energy.

The amplitude in Eq. (3.1) satisfies an integral equation

$$T(p, p'; E + i0) = U(p, p'; E + i0) - \int_0^\infty U(p, p''; E + i0) \times D(p''; E + i0)\, T(p'', p'; E + i0)\, p''^2\, dp''. \tag{3.3}$$

This equation is derived from a partial wave decomposition of Eq. (2.1a).

The effective potential U is defined by the integral

$$U(p, p'; z) = -\frac{8}{3\pi} \int_{-1}^{1} \frac{g(q_1)\, g(q_2)}{p^2 + p'^2 + pp'y - z}\, dy, \tag{3.4}$$

where

$$q_1 = (\tfrac{1}{4}p^2 + p'^2 + pp'y)^{1/2},$$
$$q_2 = (p^2 + \tfrac{1}{4}p'^2 + pp'y)^{1/2}, \tag{3.5}$$

and $g(q)$ is some smooth two-body vertex function. The momentum dependence of g depends on the particular form of the two-particle interaction. For our model calculations we choose a Yamaguchi potential [12]. The two-particle solution for this potential is described in Appendix A.

The D-function also has its origin in the solution to the two-particle problem. In our notation the two-particle d-function, defined in Appendix A, is related to the D-function in Eq. (3.3) by a shift in the three-particle energy, i.e.,

$$D(p; E + i0) = d(E + i0 - \tfrac{3}{4}p^2). \tag{3.6}$$

The evaluation of the integral in Eq. (A.6) is most conveniently done in two momentum regions; a region $p \in [0, \sqrt{\frac{4}{3}E}]$ where

$$D(p; E + i0) = -\frac{3}{4}\left(\frac{\gamma^2 + \beta^2}{C^2}\right)\left[-\beta + \left(\frac{\gamma^2 + \beta^2}{2\beta}\right) + \frac{(\gamma^2 + \beta^2)^2}{2\beta(\alpha + \beta)^2} - i\gamma\right]^{-1}, \tag{3.7}$$

and $\gamma = \sqrt{E - \frac{3}{4}p^2}$, and a region $p \in [\sqrt{\frac{4}{3}E}, \infty)$ where

$$D(p; E + i0) = -\frac{3}{4}\frac{S(\xi)}{\xi^2 - \varepsilon - i0}, \tag{3.8}$$

$$S(\xi) = \frac{(\xi + \beta)^2}{\alpha(\alpha + \beta)[1 + 2\beta/(\alpha + \xi)]}, \tag{3.9}$$

and $\xi = \sqrt{\frac{3}{4}p^2 - E}$. We mention that the D-function defined by Eq. (3.8) contains a Cauchy-type singularity $(p^2 - k^2 - i0)^{-1}$.

We now turn to the Adhikari–Sloan expansion for Eq. (3.3). From Eq. (2.3) the expansion approximation T_m reads

$$T_m(p, p'; z) = \sum_{\mu,\nu=1}^{m} \psi_\mu(p; z)\, H_{\mu\nu}(z)\, \psi_\nu(p'; z), \tag{3.10}$$

where

$$(H(z)^{-1})_{\mu\nu} = \int_0^\infty \psi_\mu(p; z)\, f_\nu(p)\, p^2\, dp + \int_0^\infty \psi_\mu(p; z)\, D(p; z)\, \psi_\nu(p; z)\, p^2\, dp. \tag{3.11}$$

The expansion functions $\{\psi_\mu; \mu = 1,..., m\}$ are given by the expression

$$\psi_\mu(p; z) = \int_0^\infty U(p, p'; z)\, f_\mu(p')\, p'^2\, dp'. \tag{3.12}$$

We see from Eq. (3.12) that ψ_μ contains an explicit p and z dependence of the exact effective potential U. The integral in Eq. (3.4) has logarithmic singularities. As z approaches the real positive energy axis, $z = E + i0$, the position of these singularities is determined by the roots of the equation

$$p^2 + p'^2 \pm pp' - E = 0. \tag{3.13}$$

Larson and Hetherington [13] have shown that these singularities pinch the p-variable integration contour along the real axis to produce a cusp in the three-body half-shell amplitude. It can be shown [14] that this cusp results from integration over the logarithmic singularities in the effective potential $U(p, p'; E + i0)$ for $E > 0$. For this reason the same cusp behaviour found in the three-body half-shell amplitude is also found in Eq. (3.12) for $\psi_\mu(p; E + i0)$ at $p = \sqrt{\frac{4}{3}E}$. We shall show later on in our numerical results that the separable expansion clearly exhibits this cusp behaviour.

The smooth functions $\{f_\mu; \mu = 1,..., m\}$ are obtained from solutions of Eq. (2.12). In its integral form this equation reads

$$-K\phi(p) = \int_0^\infty U(p, p'; -b)\, D(p'; -b)\, \phi(p')\, p'^2\, dp'. \tag{3.14}$$

The EDPE choice is now given by

$$f_\mu(p) = D(p; -b)\, \phi_\mu(p), \tag{3.15}$$

where $\phi_\mu(p)$ is a solution to Eq. (3.14).

IV. Numerical Treatment of the Separable Expansion

This section describes our numerical treatment of the separable expansion. Our method is based on the use of cubic splines.

We begin by mapping the momentum variable $p \in [0, \infty)$ onto a finite interval $[-1, +1]$. For this we use the transformation

$$p(x) = \left(\frac{1+x}{1-x}\right), \qquad x \in [-1, +1]. \tag{4.1}$$

We partition the interval $[-1, +1]$ by the knots $-1 = x_1 < x_2 < \cdots < x_n = +1$, with mesh spacing $h_n = \max\{(x_{i+1} - x_i): 1 \leqslant i < n\}$. On this partition, together with the extended knots $x_{-2} \leqslant x_{-1} \leqslant x_0 \leqslant x_1$ and $x_n \leqslant x_{n+1} \leqslant x_{n+2} \leqslant x_{n+3}$, we use a procedure given by Cox [11] to construct cubic functions $\{B_{ni}: i = 0,..., n+1\}$. These functions provide a basis for the cubic B-splines.

As an approximation to the solution of Eq. (3.14) we construct the cubic spline functions

$$\tilde{\phi}_\mu(p(x)) = \sum_{i=0}^{n+1} \alpha_{\mu,i} B_{ni}(x). \tag{4.2}$$

The expansion coefficients $\{\alpha_{\mu,i}; i = 0,..., n+1\}$ are obtained by the Galerkin method [14, 15]. A brief description of this method is outlined in Appendix B.

Since the kernel in Eq. (3.14) is continuous, then from the general properties of cubic splines it is known [16] that in the $L2$-norm the cubic spline approximation $\tilde{\phi}_\mu$ converges to ϕ_μ at a rate of $O(h_n^4)$.

We now turn our attention to calculating the functions $\{\psi_\mu; \mu = 1,..., m\}$. Let $\tilde{\psi}$ be the approximation to ψ, given by Eq. (3.12), after using the spline approximation. We write

$$\tilde{\psi}_\mu(p; E + i0) = \sum_{i=0}^{n+1} \alpha_{\mu,i} A_{ni}(p; E + i0), \tag{4.3}$$

where

$$A_{ni}(p; E + i0) = \int_0^\infty U(p, p'; E + i0)\, D(p'; -b)\, B_{ni}(x'(p'))\, p'^2\, dp'. \tag{4.4}$$

It now remains to construct the matrix $H_{\mu\nu}$. On using the approximate functions of Eqs. (4.2) and (4.3) in Eq. (3.11) we obtain

$$(H(E + i0)^{-1})_{\mu\nu} = \sum_{i,j=0}^{n+1} \alpha_{\mu,i} \alpha_{\nu,j} C_{ij}(E + i0), \tag{4.5}$$

where

$$C_{ij}(E+i0) = \int_0^\infty A_{ni}(p; E+i0)\, D(p; -b)\, B_{nj}(x(p))\, p^2\, dp + \int_0^\infty A_{ni}(p; E+i0)\, D(p; E+i0)\, A_{nj}(p; E+i0)\, p^2\, dp. \quad (4.6)$$

We note that no μ dependence is contained in the integrals of Eqs. (4.4) and (4.6). In the case when the function D contains a Cauchy principal value type singularity (see Eq. (3.8)) the integral is evaluated by the method of subtracting the singularity [17].

V. Results and Conclusions

This section describes our numerical results for the three-body separable expansion. We distinguish between two kinds of approximation: an approximation to the three-body amplitude by introducing a separable expansion, and second an approximation brought about by using numerical quadratures and splines described in the previous section. For the latter approximation we have found stable results by increasing the number of mesh points and knots.

We recall from the statement following Eq. (2.11) that the operator $T_m(z)$ is exact when operating on any linear combination of the functions $\{h_\mu; \mu = 1,..., m\}$. This does not mean, however, that the expansion approximation T_m will lead to a good pointwise agreement with the exact three-body amplitude. Indeed, the off-energy-shell three-body amplitude contains logarithmic singularities in both momentum variables p and p' arising from the inhomogeneous term in Eq. (3.3), while the expansion approximation defined by Eq. (3.10) clearly does not contain these singularities. For our separable expansion this means that increasing the number of Sturmian expansion terms m will not improve the pointwise agreement between T_m and T. On the other hand, the removal of singularities from the three-body amplitude in the Adhikari–Sloan expansion is an obvious numerical simplification of the problem.

In order to better understand this point we can consider the product integral

$$F = \int u(s)\, v(s)\, ds. \quad (5.1)$$

Here u is a singular function and v is smooth. Given that u is an integrable function one can find another function $\tilde{u} \neq u$ which replacing u in Eq. (5.1) yields the same result, that is,

$$\int (u(s) - \tilde{u}(s))\, v(s)\, ds = 0. \quad (5.2)$$

In the Adhikari–Sloan expansion the analog of $\tilde{u}$ is the function T_m.

We now give the parameters used in the three-body problem. We set $\hbar^2/M = 41.47$ MeV $-fm^2$. The two-particle binding energy $\varepsilon = 2.2267$ MeV and the range parameter $\beta = 1.44401$ fm^{-1} are chosen so that the Yamaguchi potential will approximately describe low-energy neutron–proton scattering in the S-wave spin-1 channel.

For a system of identical bosons the three-body problem gives rise to two bound states; a ground state with energy $b_1 = 25.427$ MeV and a second bound state with energy $b_2 = 2.379$ MeV. The presence of two bound states gives some flexibility in the choice of Sturmian functions. We label the solution obtained using Sturmians evaluated at $b = b_1$ by (A) and at $b = b_2$ by (B).

In spite of the fact that we do not expect a pointwise agreement between T_m and T, it is still interesting to see a numerical comparison between the two functions. For this comparison we have obtained a reference solution to the three-body amplitude by solving Eq. (2.1) using the Padé method [18]. At a three-body scattering energy $E = 4\varepsilon$ we use a [6,6] Padé approximant. In this approximation the on-shell transition amplitude T is converged to better than one part in 10^5. In order to check that our reference solution is correct a comparison is made with the reference solution used by Fonseca *et al.* [7], where agreement is found for the phase shift, δ, and inelasticity, η, over the range of energies considered by them. A further check is provided by comparing our reference solution against a solution obtained from the Doleschall [19] code.

We now describe the numerical procedure for evaluating integrals over the spline functions. In all our calculations the knots are spaced uniformly on the interval $[-1, +1]$ with mesh spacing $h_n = 2/(n-1)$. The moment integrals in Eq. (4.6) are evaluated by using a standard Gauss–Legendre quadrature formula. The actual moments are calculated over the interval $[x_i, x_{i+1}]$ and summed. An eight-point quadrature is used to integrate over smooth integrands, while for integrands containing logirithmic singularities we again split up the region of integration and evaluate the integrals with a 24-point quadrature formula.

Our first numerical results concern the eigenvalues K_μ of the Sturmian functions defined by the solution of Eq. (3.14). Table I shows the largest approximate eigenvalues obtained using the spline-Galerkin method described in Appendix B. An indication of how accurately the eigenvalues are calculated in the spline approximation is seen by comparing solutions with 24 and 30 spline basis functions. In our numerical calculations of the Sturmian expansion the eigenvalues are ordered in absolute magnitude, i.e., $|K_{\mu+1}| < |K_\mu|, \mu = 1, \ldots, m-1$. (No degenerate eigenvalues were found.)

Next we give our numerical results for the scattering sector below three-body breakup threshold. In this energy region the three-body amplitude does not contain logarithmic singularities and here we may expect that a pointwise agreement between T_m and T is possible. Table II shows the phase shift, δ, at a scattering energy $E = -0.2\varepsilon$, for several values of m. To obtain these results we have used 24 spline basis functions. As we increase the number of Sturmian expansion terms the phase shift for both choices (A) and (B) approach the Padé result. We look at the $L2$-norm to test for convergence of the half-shell function $T_m(p, k; E + i0)$. We define the norm $\|T\|_2$ as

$$\|T\|_2 = \left(\int_0^\infty |T(p, k; E + i0)|^2 \, p^2 \, dp \right)^{1/2}. \tag{5.3}$$

TABLE I

Approximate Largest Eigenvalues K_μ of the Sturmian Functions Evaluated with 24 and 30 Spline Basis Functions

(A)		(B)	
24	30	24	30
0.100(1)	0.100(1)	0.385(1)	0.385(1)
0.255	0.255	0.100(1)	0.100(1)
0.817(−1)	0.817(−1)	0.350	0.350
0.287(−1)	0.287(−1)	0.141	0.141
0.103(−1)	0.103(−1)	0.611(−1)	0.611(−1)
0.357(−2)	0.357(−2)	0.275(−1)	0.275(−1)
0.115(−2)	0.117(−2)	0.126(−1)	0.126(−1)
0.337(−3)	0.343(−3)	0.570(−2)	0.570(−2)
0.939(−4)	0.117(−3)	0.255(−2)	0.255(−2)
0.305(−4)	0.996(−4)	0.110(−2)	0.111(−2)
⋮	⋮	⋮	⋮
−0.461(−5)	−0.375(−4)	−0.481(−6)	−0.221(−4)
−0.123(−2)	−0.150(−2)	−0.117(−2)	−0.145(−2)
−0.159(−2)	−0.164(−2)	−0.155(−2)	−0.161(−2)

Table III shows the $L2$-norm of the error function $\|T_m - T\|_2/\|T\|_2$ for several Sturmian functions m and spline basis functions $n + 2$. For the values of m shown in Table III we find a stable solution with about 16 spline basis functions. As the number of Sturmian expansion terms is increased the approximation T_m again approaches the Padé result. For a given m the choice (B) gives better agreement than choice (A). This is probably because the second bound state pole dominates the low-energy scattering behaviour.

Next we consider scattering above the breakup threshold. Here, because the breakup channel has opened up, the phase shift, $\delta(k)$, is complex. The amount of absorpion into the breakup channel is measured by an inelasticity parameter

TABLE II

Solution for m Expansion Terms at Energy $E = -0.2\varepsilon$

m	(A)	(B)
2	165.5	189.5
4	210.0	215.0
6	212.0	217.6
8	212.5	218.3
Padé	219.0	

TABLE III

$L2$-Norm of the Error Function $\|T_m - T\|_2/\|T\|$ for m Expansion Terms at Energy $E = -0.2\varepsilon$

		(A)			(B)		
m	$n+2$	8	16	24	8	16	24
2		0.974	0.975	0.975	0.563	0.564	0.564
4		0.139	0.165	0.167	0.087	0.093	0.094
6		0.038	0.138	0.143	0.027	0.029	0.029
8		0.022	0.120	0.122	0.013	0.014	0.014

$\eta = \exp[-2\,\mathrm{Imag}(\delta)]$. A unitarity constraint on this parameter is that $\eta \leqslant 1$. In the case of the separable expansion this unitarity constraint may not be satisfied because the operator U_m defined by Eq. (2.2) does not contain the correct three-particle cut.

Table IV shows the real part of δ, and η, at a scattering energy $E = 4\varepsilon$. For these results we used 30 spline basis functions. Our results show that even with $m = 6$ the separable expansion solution can still be far from the correct three-body solution. This result is not surprising in view of the above comments.

In general our results for choice (B) agree with those found by Fonseca *et al.* [7]. The only discrepancy occurs for $m = 6$ at energies above breakup threshold. Here we find a smaller value for the real part of the phase shift.

Table V shows the $L2$-norm of the error function $\|T_m - T\|_2/\|T\|_2$ for values of m up to $m = 8$. Again we show results for several choices of the spline basis functions $n + 2$. In the $L2$-norm, choice (A) approaches the Padé result, but choice (B) shows no such convergence. Indeed, choice (B) gives a worse result as the number of expansion terms is increased.

TABLE IV

Solution for m Expansion Terms at Energy $E = 4\varepsilon$

	(A)		(B)	
m	Real (δ) (°)	η	Real (δ) (°)	η
2	151.4	1.52	109.8	11.9
4	135.9	2.43	155.2	11.1
6	42.1	0.26	38.2	16.3
Padé	160.3	0.813		

TABLE V

$L2$-Norm of the Error Function $\|T_m - T\|_2/\|T\|_2$ for m Expansion Terms at Energy $E = 4\varepsilon$

		(A)			(B)		
m	$n+2$	20	24	30	20	24	30
1		1.235	1.235	1.235	3.18	3.18	3.18
2		0.841	0.841	0.841	5.42	5.42	5.42
3		0.979	0.979	0.979	4.86	4.86	4.86
4		0.953	0.953	0.953	7.20	7.18	7.17
5		0.786	0.788	0.788	10.48	10.48	10.45
6		0.632	0.632	0.632	17.83	17.66	17.87
7		0.673	0.585	0.662	29.50	25.92	26.59
8		0.520	0.652	0.596	26.89	38.31	32.92

Of more practical interest in four-body calculations is the momentum structure of the expansion function T_m. Figure 1 illustrates the half-shell amplitude, $T(p, k; E + i0)$, at an energy $E = 4\varepsilon$: this function has a cusp at $p = \sqrt{\frac{4}{3}E} = 0.535\,\text{fm}^{-1}$. Figure 2 shows the results for choice (A) using $m = 6$ terms of the expansion, and Figure 3 shows the corresponding result for choice (B).

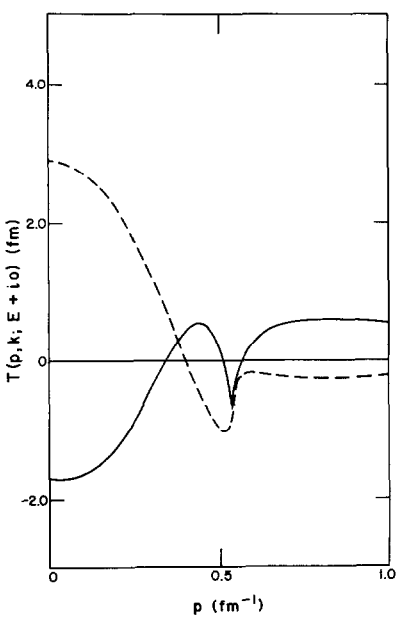

FIG. 1. Converged solution of the Padé method at an energy $E = 4\varepsilon$. Continuous and broken curves denote the real and imaginary parts of the half-shell transition amplitude.

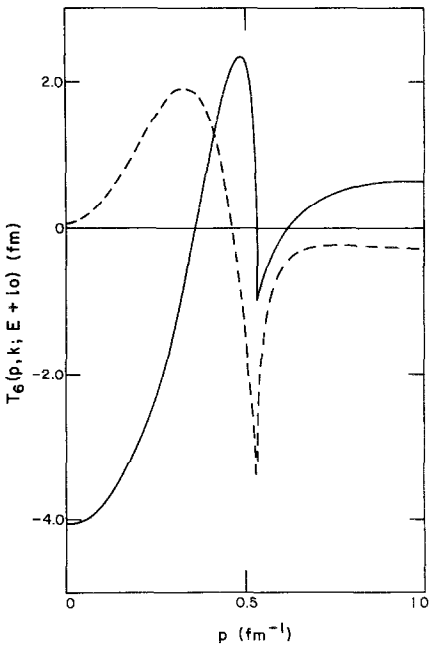

FIG. 2. Separable expansion solution with $m = 6$ at an energy $E = 4\varepsilon$. Sturmian choice (A). Curves are labelled the same as in Fig. 1.

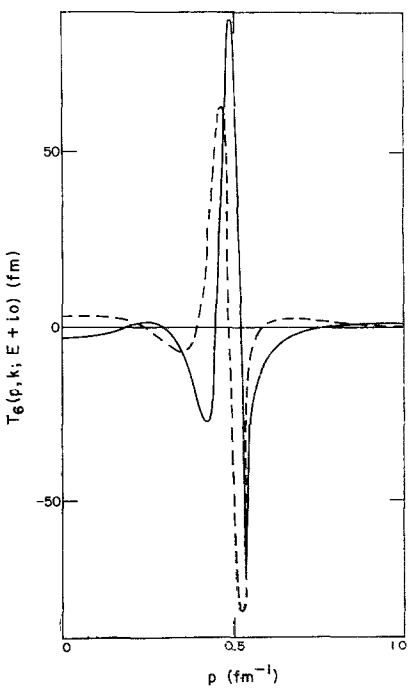

FIG. 3. Same as in Fig. 2 but with the Sturmain choice (B).

Both choices (A) and (B) display the cusp at $p = 0.535\ \text{fm}^{-1}$. However, choice (B) leads to a result having large oscillations along the p-variable. Note that the scale of Figure 3 is different from that used in the previous two figures. The origin of these oscillations is unclear, but since the idea is to use the separable expansion in the kernel of integral equations for four-body scattering, such large oscillations are an undesirable feature. In this case it would seem that the result (A) may prove a better choice. Here the approximation T_m reproduces at least the gross features of the half-shell three-body amplitude.

To summarize, we have found good pointwise agreement between the separable expansion and the three-body amplitude at energies below the breakup threshold, whereas above the breakup threshold this agreement is poor. Because no pointwise agreement between T_m and T is possible above the breakup threshold we cannot conclude that the expansion is justified in this energy region. Nevertheless, in a numerical treatment of four-body scattering the expansion function T_m is convoluted with other functions and here it is hoped that the approximation will prove useful. An application of this approach to four-body calculations is currently under investigation.

APPENDIX A: TWO-PARTICLE SOLUTION AND THE d-FUNCTION

The s-wave projection of the off-energy-shell two-particle amplitude $t(q, q'; z)$ satisfies the integral equation

$$t(q, q'; z) = v(q, q') - \frac{2}{\pi}\int_0^\infty v(q, q'')\frac{q''^2\, dq''}{q''^2 - z}\, t(q'', q'; z), \tag{A.1}$$

where q is the magnitude of the relative momentum between the two particles and z is a complex energy.

For a Yamaguchi potential [13]

$$v(q, q') = \lambda(q^2 + \beta^2)^{-1}(q'^2 + \beta^2)^{-1}, \tag{A.2}$$

where λ and β are a strength and range parameter, the solution to Eq. (A.1) can be written

$$t(q, q'; z) = g(q)\, d(z)\, g(q'). \tag{A.3}$$

Here the two-particle vertex function $g(q)$ is given by

$$g(q) = C(q^2 + \beta^2)^{-1}, \tag{A.4}$$

where C is chosen to normalize the bound state wavefunction to unity. The constant C is defined by

$$C = [2\alpha\beta(\alpha + \beta)^3]^{1/2}, \tag{A.5}$$

where $\alpha = \sqrt{\varepsilon}$, and ε is the two-particle binding energy. The d-function is given by

$$d(z) = C^{-2}\left[\frac{1}{\lambda} + \frac{2}{\pi}\int_0^\infty \frac{q^2\,dq}{(q^2+\beta^2)^2(q^2-z)}\right]^{-1}. \tag{A.6}$$

A bound state pole at $d(-\varepsilon)$ gives $\lambda = -2\beta(\beta+\alpha)^2$.

APPENDIX B: GALERKIN METHOD FOR SOLVING EQ. (3.14)

Consider the function

$$r = \sum_{i=0}^{n+1} \alpha_i [K\cdot 1 + U(-b)\,D(-b)]\,B_{ni}. \tag{B.1}$$

The Galerkin method supposes that we have defined on $C[-1, 1]$ the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)\,v(x)\,dx. \tag{B.2}$$

We now seek the coefficients $\{\alpha_i; i = 0,\ldots, n+1\}$ by solving the system of linear equations

$$(r, B_{nj}) = 0, \qquad j = 0,\ldots, n+1. \tag{B.3}$$

After rearrangement Eq. (B.3) reads

$$-K\cdot\alpha_i = \sum_{j,k=0}^{n+1} [(B_{ni}, B_{nj})]^{-1}\,(U(-b)\,D(-b)\,B_{nk}, B_{nj})\,\alpha_k. \tag{B.4}$$

In general there is a sequence of $n+2$ numbers $\{K_l; l = 0,\ldots, n+1\}$ that yield a solution to Eq. (B.4). Given that $n > m-1$, we are interested in the coefficients $\{\alpha_{\mu,i}; i = 0,\ldots, n+1; \mu = 1,\ldots, m\}$ corresponding to the m largest eigenvalues $\{K_\mu; \mu = 1,\ldots, m\}$. These coefficients are used to construct the spline functions of Eq. (4.2).

ACKNOWLEDGMENTS

The author would like to thank J.P. Svenne and E.F. Redish for helpful discussions, and also H. Fiedeldey and S. Sofianos for sustained interest in this work.

REFERENCES

1. L. D. FADDEEV, "Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory," Steklov Math. Institute 69, 1963 (Davey, New York, 1965).
2. O. A. YAKUBOVSKII, *Soviet J. Nuclear Phys.* **5** (1967), 937.
3. E. O. ALT, P. GRASSBERGER, AND W. SANDHAS, *Phys. Rev. C* **1** (1970), 85.
4. R. D. AMADO, *Ann. Rev. Nuclear Sci.* **19** (1969), 61; C. LOVELACE, *Phys. Rev. C* **1** (1970), 85.
5. For a recent review of the Hilbert–Schmidt reduction of four-particle equations see I. M. NARODETSKII, *Rev. Nuovo Cimento* **4**(7) (1981).
6. S. SOFIANOS, N. J. MCGURK, AND H. FIEDELDEY, *Nuclear Phys. A* **318** (1979), 295; S. SOFIANOS, H. FIEDELDEY, AND H. HABERZETTL, *Phys. Rev. C* **22** (1980), 1772.
7. A. C. FONSECA, H. HABERZETTL, AND E. CRAVO, *Phys. Rev. C* **27** (1983), 939.
8. F. SOHRE AND H. ZEIGELMANN, *Phys. Lett. B* **34** (1971), 579.
9. S. K. ADHIKARI AND I. H. SLOAN, *Nuclear Phys. A* **241** (1975), 429.
10. A. T. STELBOVICS, *Nuclear Phys. A* **228** (1977), 461.
11. M. G. COX, *J. Inst. Math. Appl.* **10** (1972), 134.
12. Y. YAMAGUCHI, *Phys. Rev.* **95** (1954), 1628.
13. N. M. LARSON AND J. H. HETHERINGTON, *Phys. Rev. C* **9** (1974), 699.
14. A. ALAYLIOGLU, D. EYRE, M. BRANNIGAN, AND J. P. SVENNE, submitted.
15. M. BRANNIGAN AND D. EYRE, *J. Math. Phys.* **24** (1983), 1548; **E25** (1984), 187.
16. E. ATKINSON, "A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind," SIAM, Philadelphia, 1976.
17. B. NOBLE AND S. BEIGHTON, *J. Inst. Math. Appl.* **26** (1980), 431.
18. J. A. TJON, *Phys. Rev. D* **1** (1970), 2109.
19. P. DOLESCHALL, *Nuclear Phys. A* **201** (1973), 264.